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FUNCTIONS OF AN n-DIMENSIONAL BROWNIAN MOTION THAT ARE MARKOVIAN

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ABSTRACT
Let f be a continuous function from R^n to R and let $X(t) = (X_1(t), \cdots, X_n(t))$ be a Brownian motion on R^n. The explicit form of f necessary in order to make $f(X(t))$ a Markov process is determined.

1. Introduction

Let f be a continuous function from R^n to R and $X(t)$ be a standard Brownian motion on R^n. Then what is the explicit form of f necessary in order to make $f(X(t))$ a Markov process? When $n = 1$, the following elegant result was obtained by Walsh [6].

PROPOSITION 1 (Walsh). *Let $\bar{f}$ be a continuous function from R to R and $X(t)$ be a one-dimensional Brownian motion. Let $l_0(x) = 0$, $l_1(x) = x$, $l_2(x) = |x|$ and $l_3(x) = \inf\{|x - y| \mid y$ is an even integer$\}$. Then $f(X(t))$ is a Markov process if and only if for some i, $0 \leqq i \leqq 3$,*

$$\bar{f}(x) = g \circ l_i(\alpha x + \beta), \tag{1}$$

where α and β are constants and g is continuous and strictly monotone.

In this paper, we shall extend Walsh's result to the n-dimensional Brownian motion. Our main result is given in the following.

MAIN THEOREM. *Let f be a continuous function from R^n to R and $X(t) = (X_1(t), X_2(t), \cdots, X_n(t))$ be an n-dimensional Brownian motion. Then $f(X(t))$ is Markov if and only if either*

$$f(x) = f(x_1, x_2, \cdots, x_n) = \bar{f}\left(\sum_{i=1}^{n} a_i x_i\right) \tag{2}$$

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or

$$f(x)=f(x_1,x_2,\cdots,x_n)=g\left(\sum_{i=1}^{n} b_i(x_i-c_i)^2\right), \tag{3}$$

where a_i, c_i are constants, b_i equals 1 *or* 0 *with at least two b_i's nonvanishing, and $\bar{f}$ and g are given in Proposition* 1.

In other words, in order to be a Markov process $f(X(t))$ needs to be either a function of a one-dimensional Brownian motion $\Sigma_{i=1}^{n} a_i X_i(t)$ or a function of a Bessel process $\{\Sigma_{i=1}^{n} b_i(X_i(t)-c_i)^2\}^{1/2}$. Assume that $f(X(t))=h(\Sigma_{i=1}^{n} a_i X_i(t))$. Then by Proposition 1, h must be equal to $\bar{f}$, where $\bar{f}$ is given by (1).

The functions f satisfying (2) or (3) share a common geometric property: the level surfaces of f are parallel surfaces (for a rigorous definition of parallel surfaces see Nomizu [4]). To be more precise if f is of the form (2), the level surfaces of f are parallel hyperplanes; if f is of the form (3), then the level surfaces of f are the surfaces of a family of concentric spherical cylinders. In particular, when $n=3$ the level surfaces of f can be parallel planes, surfaces of concentric balls, or surfaces of concentric cylinders.

If f is given by (2) or (3), then $f(X(t))$ is clearly Markovian. Hence we only need to prove the other direction of our main theorem. From now on let $f(X(t))$ be Markovian. In section 2 we shall obtain some geometric properties that f satisfies. One of the properties is:

$$f(x)=f(x') \quad \text{implies} \quad d(x,f^{-1}(c))=d(x',f^{-1}(c)) \tag{4}$$

where $\{x,x'\}\subset R^n$, $c\in R$, and d is the distance function. In other words, points of equal value in f are equidistant from any level set of f. This is very close to saying that the level surfaces of f are parallel. Since f is only continuous, in the beginning we do not know if the level sets are surfaces. Hence a transformation of f is needed. In section 3, we prove that $f=u\circ h$, where h is a function from R^n to R such that it is harmonic in $h^{-1}(\text{Interior } h(R^n))$ and u is continuous and strictly monotone from $h(R^n)$ to R. Since $h=u^{-1}\circ f$ and u^{-1} is strictly monotone, h is also a Markovian function. Further, if one can prove that h is of the form given by either (2) or (3), then so is f.

In our initial approach to this problem, we only used the smoothness of h and the fact that h satisfies (4) to prove our main theorem. However, the proof was fairly complicated. Later, we found that on $h^{-1}(I^\circ)$, $I^\circ=$ interior of $h(R^n)$, h satisfies a system of partial differential euations:

There exist Borel measurable functions ϕ and ψ such that

$$(5) \qquad |\operatorname{grad} h|(x) = \phi(h(x)) \quad \text{and} \quad \Delta h(x) = \psi(h(x)).$$

Equation (5) was studied by E. Cartan, K. Nomizu and others. For an elegant treatment of (5) and some of the references see Nomizu [4]. Using the results in [4], it is easy to see that $h(x_1, \cdots, x_n)$ must be equal to either $g_1(\Sigma_{i=1}^n a_i x_i)$ or $g_2(\Sigma_{i=1}^n b_i (x - c_i)^2)$ for some g_1 and g_2, where a_i, b_i and c_i are given in the main theorem. Then it is not hard to determine g_1 und g_2.

A few more references related to our problem need to be mentioned. Rosenblatt [5] studied the Markovian functions f of a stationary Markov chain X_n. Under the hypothesis that the transition probabilities of X_n are dominated by a sigma finite measure, he obtained some necessary and sufficient conditions on f to make $f(X_n)$ Markovian. For a stationary Markov process $Z(t)$ Dynkin [2; p. 325] has a sufficient condition on f to make $f(Z(t))$ a Markov process. In both [5] and [2], their processes are defined on a general state space and the conditions they obtained are in terms of the transition functions. The explicit form of Markovian functions of a general Markov process are hard to find. When the process is a one-dimensional space–time Brownian motion or one-dimensional homogeneous diffusion, the explicit forms of Markovian functions are given by Wang [7] and [8] respectively.

2. Some geometric properties of a Markovian function of Brownian motion

Let $(X(t), \mathcal{F}_t, P_x)$ be a standard Brownian motion and $p(x, t, A)$, $x \in R^n$, $t \geqq 0$, $A \in \mathcal{B}(R^n)$, be its transition function. Let f be a continuous function from R^n to R, $Y(t) = f(X(t))$, and $\mathcal{G}_t$ be the sigma field generated by $\{Y(u) \mid u \leqq t\}$. By calling $Y(t)$ a Markov process, one usually means that $\{Y(t), \mathcal{G}_t, P_x\}$ is Markovian. Indeed $\{Y(t), \mathcal{G}_t, P_x\}$ is a Markov process if and only if $\{Y(t), \mathcal{F}_t, P_x\}$ is Markovian (see Wang [8]). Hence there is no ambiguity when one simply says that $Y(t)$ is a Markov process. The following class T of functions are exactly the continuous Markovian functions of Brownian motion: $f \in T$ if f is a continuous function from R^n to R such that

$$(6) \quad f(x) = f(x') \text{ implies } p(x, t, f^{-1}(B)) = p(x', t, f^{-1}(B)), \; B \in \mathcal{B}(R), \; t \geqq 0.$$

LEMMA 1. *$f(X(t))$ is Markovian if and only if $f \in T$. Further, if $f \in T$ then $f(X(t))$ is strong Markov with respect to both $\mathcal{F}_t$ and $\mathcal{G}_t$.*

A proof of Lemma 1 can be found in [8] and hence is omitted. The idea of this lemma goes back to Rosenblatt [5] and Dynkin [2; theorem 10.13].

Let $z \in R^n$ and $S(z, r)$ be the surface of the ball centered at z with radius r.

For $r \geqq 0$, $B \in \mathscr{B}(R)$ we define

$$G(z, r, f^{-1}(B)) = \text{surface measure of } S(z, r) \cap f^{-1}(B). \tag{7}$$

Assume $f \in T$. We shall obtain:

$$f(x) = f(x') \quad \text{implies} \quad G(x, r, f^{-1}(B)) = G(x', r, f^{-1}(B)), \quad B \in \mathscr{B}(R), \tag{8}$$

for almost all r.

Indeed, f satisfies (8) is equivalent to $f \in T$. An important consequence of (8) is (4). We shall prove these results in the following.

LEMMA 2. *Let $v \in L^1_{\mathrm{loc}}([0, \infty))$ and let*

$$\int_0^\infty \exp[-r^2/(2t)]v(r)dr = 0 \qquad \textit{for all } t > 0.$$

Then $v(r) = 0$ a.e.

PROOF. Put $u = 1/(2t)$, $z = r^2$. We get

$$\int_0^\infty \exp(-uz)d\alpha(z) = 0 \qquad \text{for all } u,$$

where $d\alpha(z) = v(z^{1/2})(4z)^{-1/2}dz$ is a function of bounded variation on any compact subset *of* $[0, \infty)$. *By Widder* [9; *p.* 107 *theorem* 7.2], $\alpha(z) = 0$ for all $z \in [0, \infty)$. This implies $v(r) = 0$ a.e.

LEMMA 3. *$f \in T$ if and only if f satisfies* (8).

PROOF.

$$p(z, t, f^{-1}(B)) = (2\pi t)^{-n/2} \int_{f^{-1}(B)} \exp[-|z - y|^2/(2t)]dy$$

$$= (2\pi t)^{-n/2} \int_0^\infty \exp[-r^2/(2t)] \cdot G(z, r, f^{-1}(B))dr.$$

Hence for any two points $\{x, x'\} \subseteq R^n$, $p(x, t, f^{-1}(B)) = p(x', t, f^{-1}(B))$ if and only if

$$\int_0^\infty \exp[-r^2/(2t)] \cdot [G(x, r, f^{-1}(B)) - G(x', r, f^{-1}(B))]dr = 0.$$

Then Lemma 3 follows easily from Lemma 2.

COROLLARY 1. *Let $f \in T$ and $f(x) = f(x')$. Then $d(x, f^{-1}(B)) = d(x', f^{-1}(B))$ for any open subset B of R such that $f^{-1}(B) \neq \varnothing$.*

PROOF. Let $d(x, f^{-1}(B)) = d_0$. Given any $\varepsilon > 0$, there exists $y \in f^{-1}(B)$ such that $d(x, y) < d_0 + \varepsilon$. Since $f^{-1}(B)$ is an open set, there is a $\delta > 0$ such that the ball centered at y with radius δ is contained in $f^{-1}(B)$. Hence

$$G(x, r, f^{-1}(B)) > 0 \qquad \text{for a.a. } r \in J = [d(x, y) - \delta, d(x, y) + \delta].$$

By Lemma 3 and the assumption that $f(x) = f(x')$, we have $G(x', r, f^{-1}(B)) > 0$ for a.a. $r \in J$. Hence $d(x', f^{-1}(B)) < d(x, y) < d_0 + \varepsilon$. Then Corollary 1 follows easily.

COROLLARY 2. *Let f be a nonconstant function in T. Then $f^{-1}(a)$ does not have an interior point for any $a \in R$.*

PROOF. Assume for some a, $f^{-1}(a)$ has an interior point x. Since f is a nonconstant continuous function, $f^{-1}(a)$ has a boundary point $x' \in f^{-1}(a)$. Let $B = R - \{a\}$ in Corollary 1, one gets

$$d(x, f^{-1}(B)) = d(x', f^{-1}(B)) = 0.$$

This contradicts our assumption that x is an interior point of $f^{-1}(a)$.

Similarly, one can also show that for a nonconstant function $f \in T$, $f^{-1}(a)$ does not contain a density point for any $a \in R$.

COROLLARY 3. *If $f \in T$, then f satisfies* (4).

PROOF. If $f^{-1}(a) = \varnothing$, then (4) is clearly true. Assume $f^{-1}(a) \neq \varnothing$. Then Corollary 3 follows from Corollary 2, because

$$d(x, f^{-1}(a)) = \lim_{\varepsilon \to 0} d(x, f^{-1}(a - \varepsilon, a + \varepsilon)).$$

REMARK 1. Is the converse of Corollary 3 true? The answer is negative Let C_s, $s \geqq 0$, be a collection of subsets of R_2 such that

$$C_0 = \{(x, y) \mid x \in [-1, 1], y = 0\},$$

$$C_s = \{(x, y) \mid d((x, y), C_0) = s\}.$$

Define $u(x, y) = s$ for $(x, y) \in C_s$. Clearly, u satisfies (4). But u is not of the form given in the main theorem.

REMARK 2. The results we found in this section are also true when f is a function from R^n to R^m, $m > 1$.

REMARK 3. If one allows f to be a continuous function from R^n to $[-\infty, \infty]$, all results in this section still hold, presuming one also modifies the definition of T accordingly.

3. Some analytic properties of Markovian functions of Brownian motion

Let $f \in T$ and let $Y(t) = f(X(t))$. Then $Y(t)$ is a pathwise continuous process. For $y = f(x)$, we use Q_y to denote the P_x-distribution of $Y(t)$. By the definition of T, Q_y is well defined. Put

$$a = \inf\{f(x) \mid x \in R^n\}, \qquad b = \sup\{f(x) \mid x \in R^n\}.$$

We shall assume $a < b$, i.e. f is a nonconstant function. By Lemma 1, $\{Y(t), \mathcal{F}_t, Q_y\}$ is a diffusion taking values from $[a, b]$. We discuss the regularity of $\{Q_y\}$ in the following.

Let $\{y, z\} \subseteq (a, b)$. Without loss of generality we can assume $y < z$. Then

$$\begin{aligned} Q_y(Y(t) \text{ hits } z) &\geqq Q_y(Y(t) \text{ hits } (z, b)) \\ &= P_x(X(t) \text{ hits } f^{-1}(z, b)) \\ &> 0, \qquad \text{where } f(x) = y. \end{aligned}$$

If both a and b are not taken by f, we have already shown that $\{Q_y\}$ is a regular diffusion on (a, b). If one or both of $\{a, b\}$ are taken by f, then more discussions are needed. Assume a is taken by f and $y \in (a, b)$. Then

$$Q_y(Y(t) \text{ hits } a) = P_x(X(t) \text{ hits } f^{-1}(a)) \quad \text{for some } x \text{ such that } f(x) = y.$$

Clearly, $P_x(X(t) \text{ hits } f^{-1}(a)) > 0$ if $f^{-1}(a)$ is not a polar set. On the other hand if $f^{-1}(a)$ is a polar set, then we can ignore it and treat the process $\{f(X(t)), \mathcal{F}_t, P_x\}$ (or equivalently the process $\{Y(t), \mathcal{F}_t, Q_y\}$) as a process taking values from $(a, b]$ or (a, b).

From now on we shall assume that $\{Y(t), \mathcal{F}_t, P_x\}$ is a regular diffusion on an interval I, where I can be (a, b), $(a, b]$, $[a, b)$, or $[a, b]$. Let S be the scale function of $\{Q_y\}$, $y \in I$. Then $\{S(Y(t)), \mathcal{F}_t, Q_y\}$ is a regular diffusion of natural scale on $S(I)$. Define

$$\tau = \inf\{t \mid Y(t) = a \text{ or } b\} = \inf\{t \mid X(t) \in f^{-1}(a) \cup f^{-1}(b)\}.$$

Then it is well known that for each fixed $y \in I^\circ$, the interior of I, $\{S(Y(t \wedge \tau)), \mathcal{F}_{t\wedge\tau}, Q_y\}$ is a fair process and hence a local martingale. Equivalently, $\{S \circ f(X(t)), \mathcal{F}_{t\wedge\tau}, P_x\}$ is a local martingale for each $x \in f^{-1}(I^\circ)$. Put $D = f^{-1}(I^\circ)$. By the continuity of f, $D = \bigcup_{i=1}^{\infty} O_i$ where O_i are disjoint connected open sets.

LEMMA 4. *Let $h = S \circ f$. Then h is harmonic on each component of D.*

PROOF. Since $(h(X(t \wedge \tau)), \mathcal{F}_{t\wedge\tau}, P_x)$ is a local martingale for each $x \in D$, it follows easily that h is *harmonic for Brownian motion* on each O_i. For the definition of harmonic functions for a process, we refer the reader to Dynkin [5; 12.11]. Applying theorem 12.11 of Dynkin [5], one obtains that h is harmonic on each O_i.

The scale function S is strictly monotone on I, $(a, b) \subseteq I \subseteq f(R^n)$. If no polar set has been ignored in obtaining the regularity of $f(X(t))$, then $I = f(R^n)$. Otherwise, $I \neq f(R^n)$. Since S is monotone on I, one can extend it continuously to $[a, b]$. Then $h = S \circ f$ is defined on R^n. Note that h may be infinite on a polar set. Because S is strictly monotone and $f \in T$, h satisfies (6). By Remark 3, the results given in section 2 hold true for function h.

Before we prove that h satisfies (5) on D, we state a result given by Dynkin [2, theorem 10.13].

PROPOSITION 3. *Let $u \in T$ and g be a function from R^n to R which is measurable with respect to u. Let $T_t g(x) = E^x(g(X(t)))$ be finite for all x. Then $T_t g$ is also measurable with respect to u.*

To prove Proposition 3, one needs to show that $u(x) = u(x')$ implies $T_t g(x) = T_t g(x')$. This follows easily from the fact that u satisfies (6), $g(x) = v(u(x))$ a.e. for some Borel measurable v (see Chung [1, p. 299]), and

$$T_t g(x) = \int_{R^n} g(x) p(x, t, dy).$$

LEMMA 5. *Let $h = S \circ f$ be given as in Lemma 4. Then h satisfies (5) on D.*

PROOF. Since h is harmonic on each component of D, $\Delta h = 0$ on D. Define $\phi(x) \equiv 0$, then $\Delta h = \phi \circ h$ on D. Now let $u(x) = \tan^{-1} x$, $v(x) = u \circ h(x)$. Since u is strictly monotone and h satisfies (6), v satisfies (6). That is, $v \in T$. By using Proposition 3 and the properties of the generator of Brownian motion, it is not hard to show that Δv is measurable with respect to v and hence Δv is also measurable with respect to h. Now for $x \in D$,

$$\begin{aligned}\Delta v(x) &= u'(h(x))\Delta h(x) + u''(h(x))|\operatorname{grad} h(x)|^2 \\ &= u''(h(x))|\operatorname{grad} h(x)|^2 \\ &= w(h(x))\end{aligned}$$

for a Borel measurable function w defined on the reals. Evidently, for

$x \in D - B$, $|\operatorname{grad} h(x)|^2 = \phi(h(x))$ where $\Phi(y) = w(y)/u''(y)$, and $B = \{x \mid u''(h(x)) = 0\}$. The exceptional set B is easy to eliminate. One of the ways to eliminate it is by defining $u_1(x) = \tan^{-1}(x+1)$, $v_1(x) = u_1 \circ h(x)$ and repeating our previous arguments again.

REMARK 4. Let u be a C^2 function in T. Then by using arguments similar to that given in Lemma 5, one can prove that u satisfies (5). But there exist C^2 functions which satisfy (5) and are not in T. Let g be a C^2 function from $[0,\infty)$ to R such that g is strictly decreasing in $[0,1]$ and $g(y) = 0$ for all $y \geqq 1$. Put $l(x_1, x_2, \cdots, x_n) = g(\sum_{i=1}^n x_i^2)$, then l satisfies (5). However, since g is not strictly monotone, l is not of the form given in the main theorem. Thus $l(X(t))$ is not Markovian.

4. Proof of the main theorem

Let u be a real valued C^2 function defined on R^n. Let u satisfy (5). The following results are known (see Nomizu [4]).

(1) Put $M_s = \{x \mid u(x) = s\}$. Assume that $\operatorname{grad} u(x) \neq 0$ on a certain M_s, then M_s has constant principal curvatures.

(2) In R^n there are exactly n kinds of hypersurfaces of constant curvatures. They are the level surfaces of $w(x_1, \cdots, x_n) = \sum_{i=1}^k x_i^2$, $1 \leqq k \leqq n$.

The arguments given in Nomizu [4] are local arguments and can be applied also to the function h given in Lemma 5. By abusing our notation slightly, we shall use M_s to denote $\{x \mid h(x) = s\}$.

Let $x_0 \in D$, $h(x_0) = s$, and $\operatorname{grad} h(x_0) \neq 0$. Then the component of M_s which passes through x_0 is a hypersurface of constant curvature. By the smoothness of h, we know there is an open neighborhood N of x_0 such that $\operatorname{grad} h(x) \neq 0$ for all $x \in N$. Hence for each $x \in N$, the level surface of h which passes through x is also of constant curvature. It is easy to see that the level surfaces of h passing through $x, x \in N$, are parallel. We can expand this family of parallel surfaces until we need to cross a point x such that $x \notin D$ or $\operatorname{grad} h(x) = 0$.

Put $B_1 = \{x \in D \mid \operatorname{grad} h(x) = 0\}$, $B_2 = B_1 \cup D^c$. Since $D^c \subseteq f^{-1}(a) \cup f^{-1}(b)$ (recall $h = S \circ f$, $f \in T$) and $a < b$, by Corollary 2 we know D^c does not have an interior point. We claim B_1 does not contain an interior point either. Otherwise, $h^{-1}(c)$ would have an interior point for some c, a contradiction to Remark 3 and Corollary 2. Since $B_1 \subseteq D$ and D is an open set, one obtains that $B_2 = B_1 \cup D^c$ does not have an interior point. Now by using Corollary 3 and Remark 3, one sees that the parallel family of level surfaces of h obtained in the last paragraph can cross B_2 and be expanded to cover R^n.

Since $f = S^{-1} \circ h$ and S^{-1} is a strictly monotone function, the level surfaces of f are parallel and are of constant curvature. One can find two Borel measurable functions g_1 and g_2 such that

$$f(x_1, x_2, \cdots, x_n) = g_1\left(\sum_{i=1}^{n} a_i x_i\right), \quad \text{or}$$

(9)

$$= g_2\left(\sum_{i=1}^{n} b_i (x_i - c_i)^2\right),$$

where a_i, b_i, c_i are given in the main theorem.

Let $f(x_1, x_2, \cdots, x_n) = g_1(\Sigma_{i=1}^{n} a_i x_i)$. Since $\Sigma_{i=1}^{n} a_i X_i(t)$ is a one-dimensional Brownian motion and $f(X(t))$ is Markovian, by Proposition 1 we know $g_1 = \bar{f}$. We shall prove that the function g_2 in (9) is strictly monotone.

LEMMA 6. *Let $f \in T$, $f \not\equiv$ Const. Suppose that*

$$f(x_1, \cdots, x_n) = g\left(\sum_{i=1}^{n} b_i (x_i - c_i)^2\right),$$

where b_i, c_i are given in the main theorem, then g is strictly monotone.

PROOF. We shall only prove the case when $f(x_1, \cdots, x_n) = g(\Sigma_{i=1}^{k} x_i^2)$, where $k = 2, 3, \cdots n$. The general case can be treated similarly. For the special case we are going to deal with, we can treat f as a function defined on R^k.

Assume $g(0) = a$. First, we'll show that $g(r) \neq a$ for all $r \neq 0$. Assume this is not true so there exists at least one $r_0 \neq 0$ such that $g(r_0) = a$. From the properties of f and g and Corollary 2, we know g is not a constant function on $[0, r_0]$. Let M and m be the maximal value and the minimal value of g on $[0, r_0]$ respectively. Evidently, a is properly contained in $[m, M]$. Assume $a \neq M$. Define

$$r_1 = \inf\{r \mid g(r) = M\},$$

$$r_2 = \sup\{r \mid r < r_1, \ g(r) = (a + M)/2\}.$$

Now let $x = (x_1, \cdots, x_k)$ be a point in R^k such that $\Sigma_{i=1}^{k} x_i^2 = r_0$ and let $B = ((a + M)/2, M)$. Clearly, $f(x) = f(0)$; here 0 denotes the origin of R^k. By Lemma 3 $G(x, r, f^{-1}(B)) = G(0, r, f^{-1}(B))$ for almost all r. By the definition of r_1 and r_2, it is easy to see that $G(0, r, f^{-1}(B)) = C(k, r)$ for $r_1 < r < r_2$, where $C(k, r)$ denotes the surface measure of a k-dimensional ball with radius r. Since g is continuous and $g(r_0) = a$, there exists an $\varepsilon > 0$ such that $g(r) < (a + M)/2$ for all $r \in [r_0 - \varepsilon, r_0 + \varepsilon]$. Then by simple geometry, it is not hard to see that $G(x, r, f^{-1}(B)) < C(k, r)$ for all r in $[r_1, r_2]$. Hence $f(x) \neq a$. If $a = M$ then $a \neq m$. We can proceed similarly and obtain $f(x) \neq a$.

Since f cannot take value a at any other point in R^n, the origin is either the only global maximum of f or the only global minimum of f. Assume g is not strictly monotone. Then there exist $\{r_3, r_4\}$, $r_3 \neq r_4$, such that $g(r_3) = g(r_4)$. Let y and z be two points in R^k of distance r_3 and r_4 from the origin respectively. Then $f(y) = f(z)$, but $d(y, f^{-1}(a)) \neq d(z, f^{-1}(a))$, a contradiction to Corollary 3. Hence g must be strictly monotone.

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